4. TRIANGLES IN GRAPHS

Another stupid party with six people to look on, and talk nonsense to each other. **Jane Austen**

§4.1. Ramsay Numbers

An interesting topic of conversation at a party is the fact that for every six people there are always three people who know each other or three people who are mutual non-acquaintances. This is known as Ramsay's Problem.

A **triangle** in a graph is a subgraph that's isomorphic to K_3. That is, it is a set of 3 vertices where each is adjacent to the other two.

Theorem 1 (RAMSEY): If G is a graph with 6 vertices then either G or $\overline{G}$ contains a triangle.
Proof: Let G have 6 vertices and let V be one such vertex. For each of the other 5 vertices, either V is adjacent to at least 3 of them or it is not adjacent to at least 3 of them.

In other words, V is adjacent to at least 3 vertices in either G or $\overline{G}$.

Without loss of generality we may assume that there are vertices A, B, C adjacent to V in G. (If not we just replace G by its complement.)

If two of these are adjacent in G, say A and B, then we have a triangle VAB in G.

If no two are adjacent then ABC is a triangle in $\overline{G}$.✋☺

Another way of expressing this theorem is that every graph on 6 vertices either contains K_3 or its complement. Another way of expressing the theorem is that if each edge of the complete graph K_6 is coloured either red or green there must be a monochromatic K_3 (that is a triangle with all three edges coloured the same colour). Here we would consider the graph G to be the subgraph with all 6 vertices but only the red edges, while the complement uses the green edges instead.

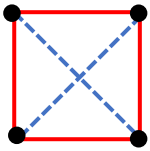
A **k-coloured graph** is a graph for which the set of edges is partitioned into subsets $C_1, C_2, \ldots, C_k$. We consider the C_i as being the colours.

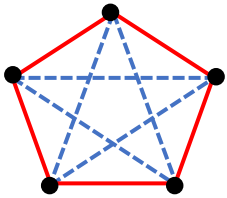
What is the smallest integer R(*m*, *n*) such that every 2-coloured graph, with colours red and blue, contains either

a red K_m or a blue K_n. These numbers are called the **Ramsey numbers**. Clearly they're symmetric in m, n.
Clearly $R(m, 1) = 1$ for all m since the graph with 1 vertex is itself K_1 and $K_1 = \overline{K}_1$
Any pair of non-adjacent vertices is isomorphic to $\overline{K}_2$ so if a graph has m vertices it is either K_m, or its has a pair of non-adjacent vertices, which is K_2 in the complement of the graph. Hence $R(m, 2) = m$ for all m.

Ramsey's Theorem shows that $R(3, 3) \leq 6$. To show that it equals 6 we need to consider graphs with fewer than 6 vertices.

Theorem 2: $R(3, 3) = 6$.
Proof: For graphs with 1 or 2 vertices, neither they nor their complement can contain a triangle. The following 2-colourings (with the solid lines representing red edges and the dashed lines representing blue ones) of K_4 and K_5 don't contain any red or blue triangles. In other words, neither the red graphs, nor their complements, contain a triangle. ✋☺





There's no known formula for the Ramsey numbers. The best we have is an upper bound, due to Erdös and Szekeres:

$$R(m, n) \le \binom{m + n - 2}{m - 1}.$$

For $m = n = 3$, the inequality is best possible.

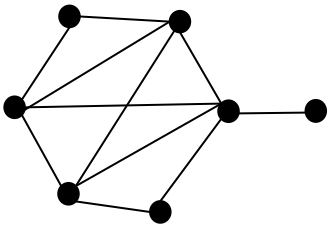
Beyond these values there's only a handful of values known. They are given in the following table.

R(*m*,*n*)	**3**	**4**	**5**	**6**	**7**	**8**	**9**
3	6	9	14	18	23	28	36
4		18	25	?	?	?	?

§2.2. Cliques

A **clique** in a graph G is a maximal complete subgraph. That is, it's a subgraph in which every vertex is adjacent to every other and where no subgraph containing it is complete.

Example 1: The graph



has one clique of size 4, 2 cliques of size 3 and one clique of size 2. Altogether it has 6 triangles, but 4 of these, the ones in the K_4 clique, are not maximal.

Now we've defined R(m, n) as the smallest number such that every graph on this number of contains either contains K_m or the complement of K_n. But is there such an integer? You can't define something to be the smallest if there aren't any! Clearly we can find a graph on $m + n$ vertices that contains *both* K_m and the complement of K_n, but what about all the other graphs on $m + n$ vertices?

One way of looking at this problem, and something that leads to a generalisation, is to consider colouring the edges of a graph. For a graph to have either K_m or the complement of K_n as a subgraph is equivalent to it being 2-coloured so that it contains a monochromatic clique of size m or its complement has a clique of size n.

Theorem 2: For all integers m, n there exists an integer R such that every graph on R vertices contains a clique of size m and its complement contains a clique of size n.

Theorem: For all integers m, n, c there exists an integer R(m, n) such that every 2-coloured graph on R vertices contains a monochromatic clique of size m or its complement contains a monochromatic clique of size n.

Proof: Induction on $m + n$.
It's clear from the definition that for all n:

$$R(n, 2) = R(2, n) = n.$$

This starts the induction.

We prove that $R(m, n)$ exists by finding an explicit bound for it. By the inductive hypothesis both $R(m - 1, n)$ and $R(m, n - 1)$ exist.

We prove that $R(m, n) \leq R(m - 1, n) + R(m, n - 1)$.
Let $T = R(m - 1, n) + R(m, n - 1)$.
Colour the edges of $G = K_T$ red and blue.
Pick a vertex V from G, and partition the remaining $T - 1$ vertices into two sets M and N as follows:
$M = \{W \mid (V, W) \text{ is blue}\}$ and $N = \{W \mid (V, W) \text{ is red}\}$.
Now $T = R(m - 1, n) + R(m, n - 1) = |M| + |N| + 1$.
Hence either $|M| \geq R(m - 1, n)$ or $|N| \geq R(m, n - 1)$.

If $|M| \geq R(m - 1, n)$ then M has a blue K_{m-1} or a red K_n.
In the former case $M \cup \{V\}$ has a blue K_m, and hence, so has G. In the latter case M has a red K_n and hence, so has G.
If $|N| \geq R(m, n - 1)$ then N has a blue K_m or a red K_{n-1}.
In the former case N has a blue K_m, and hence, so has G.
In the latter case $N \cup \{V\}$ has a red K_n and hence, so has G.

c > 2: We prove that:

$$\mathbf{R}(n_1, \ldots, n_c) \leq \mathbf{R}(n_1, \ldots, n_{c-2}, \mathbf{R}(n_{c-1}, n_c)).$$

Let $T = R(n_1, \ldots, n_{c-2}, R(n_{c-1}, n_c))$ and let $G = K_T$.
Now c-colour G.
Combine colours C_{c-1} and C_c. G is now $(c - 1)$-coloured. By induction G contains either a monochrome K_{n_i} for some i with $1 \leq i \leq c - 2$ or a monochrome K_N where $N = R(n_{c-1}, n_c)$ in combined colour.

In the former case we're finished.
In the latter case, our K_N is 2-coloured. Hence this K_N must have either a $(c - 1)$-monochrome K_S where $S = n_{c-1}$ coloured C_{c-1} or a monochrome K_Z where $Z = n_c$ coloured C_c. In either case the proof is complete.

Only two non-trivial multicolour Ramsey numbers are known, namely R(3, 3, 3) = 17 and R(3, 3, 4) = 30.

Theorem: R(3, 3, 3) = 17.
Proof: Suppose that we have an edge colouring of a complete graph using 3 colours, red, green and blue, with no monochromatic triangles.
Select a vertex V.
Consider the set of vertices that have a red edge to the vertex V. This is called the red neighbourhood of V.

The red neighbourhood of V can't contain any red edges, since otherwise there would be a red triangle.

Thus, the induced edge colouring on the red neighbourhood of V has edges coloured with only two colours, namely green and blue.

Since $R(3, 3) = 6$, the red neighbourhood of V can contain at most 5 vertices. Similarly, the green and blue neighbourhoods of V can contain at most 5 vertices each. Since every vertex, except for V itself, is in one of the red, green or blue neighbourhoods of V, the entire complete graph can have at most $1 + 5 + 5 + 5 = 16$ vertices.
Thus, we have $R(3, 3, 3) \leq 17$.

To see that $R(3, 3, 3) \geq 17$, it suffices to draw an edge colouring on the complete graph on 16 vertices with 3 colours that avoids monochromatic triangles. It turns out that there are exactly two such colourings on K_{16}.

§4.3. Triangle-Free Graphs

A **triangle-free** graph is one that has no triangles. A graph on a certain number of vertices can be triangle-free if there are very few edges. What is the largest number of edges for a triangle-free graph on a given number of vertices?

Let **D(n)** be the largest number of edges that a graph on n vertices can have and still be triangle-free.

Theorem (DIJKSTRA):

$$D(2k) = k^2 \text{ and } D(2k - 1) = k(k - 1).$$

Proof: Both equations for D have the form $M(a+b) = ab$. We show that a graph with $a + b$ vertices and ab edges needn't contain a triangle by constructing a counter-example.

Paint a of the vertices red and the remaining b vertices blue.

Connect each red vertex to each blue vertex.

Along any cycle colours must alternate and so cycles have even length and so the graph is triangle-free.

That a graph with more edges – if feasible – contains a triangle is shown by mathematical induction.
We confine our attention to graphs with at least 3 vertices, and for the base of the induction we consider

$$(a, b) = (1, 2) \text{ and } (a, b) = (2, 2).$$

In the first case, we have to show that a graph with #vertices = 1 + 2 and #edges > 1× 2 contains a triangle: such a graph has 3 vertices and 3 edges and is a triangle.

In the second case, we have to show that a graph with #vertices = 2 + 2 and #edges > 2×2 contains a triangle: such a graph has at least 5 edges and hence contains the complete 4-graph with 1 edge missing, which contains even 2 triangles.

For the induction step we show for any a, b:

if any graph with #vertices = a+b and #edges > ab
contains a triangle then any graph with
#vertices = (a+1)+(b+1) and #edges > (a+1)(b+1)
contains a triangle.

To this end we consider a graph with

(a+1)+(b+1) = a + b + 2 vertices

and more than (a+1)(b+1) = ab+a+b+1 edges

and focus our attention on one of its edges e.

Now we consider two cases:

e is or is not an edge of a triangle.

In the former case the graph obviously contains a triangle. If e is not the edge of a triangle, however, it's connected by at most a + b edges to the subgraph spanned by the remaining a + b vertices.

The total graph having more than ab + a + b + 1 edges, we conclude that in this case the subgraph of a + b vertices

has more than ab edges and, by induction, contains a triangle.

In view of the base we have demonstrated the presence of a triangle for $a + 1 = b$ or $a = b$ and $b \geq 2$.

The above proof is noteworthy for two reasons, firstly because even and odd number of vertices could share the same induction step, and secondly because the presence of a triangle in a graph with 4 vertices and 5 edges has been demonstrated without case analysis. (Take the experiment and ask your friends.)